http://www.purplemath.com/modules/factquad.htm

Factoring Quadratics: The Simple Case *(page 1 of 4)*

Sections: The simple case, The hard case, The weird case

A "quadratic" is a polynomial that looks like "$ax^2 + bx + c$", where "a", "b", and "c" are just numbers.

For the easy case of factoring, you will find two numbers that will not only multiply to equal the constant term "c", but also add up to equal "b", the coefficient on the x-term. For instance:

- **Factor $x^2 + 5x + 6$.**

  I need to find factors of 6 that add up to 5. Since 6 can be written as the product of 2 and 3, and since $2 + 3 = 5$, then I'll use 2 and 3. I know from multiplying polynomials that this quadratic is formed from multiplying two factors of the form "$(x + m)(x + n)$", for some numbers m and n. So I'll draw my parentheses, with an "x" in the front of each:

  $(x \quad)(x \quad)$

  Then I'll write in the two numbers that I found above:

  $(x + 2)(x + 3)$

  This is the answer: $x^2 + 5x + 6 =$ **$(x + 2)(x + 3)$**

This is how all of the "easy" quadratics will work: you will find factors of the constant term that add up to the middle term, and use these factors to fill in your parentheses.

Note that you can always check your work by multiplying back to get the original answer. In this case:

$$\begin{array}{r} x+2 \\ x+3 \\ \hline 3x+6 \\ x^2+2x \quad \\ \hline x^2+5x+6 \end{array}$$

Your text or teacher may refer to factoring "by grouping", which is covered in the lesson on simple factoring. In the "easy" case of factoring, using "grouping" just gives you some extra work. For instance, in the above problem, in addition to finding the factors of 6 that add to 5, you would have had to do these additional steps:

$$\begin{aligned} x^2 + 5x + 6 &= x^2 + 3x + 2x + 6 \\ &= (x^2 + 3x) + (2x + 6) \\ &= x(x + 3) + 2(x + 3) \\ &= (x + 3)(x + 2) \end{aligned}$$

You get the same answer as by the previous method, but I think it's easier to just fill in the parentheses.

- **Factor $x^2 + 7x + 6$.**

  The constant term is 6, which can be written as the product of 2 and 3 or of 1 and 6. But 2 + 3 = 5, so 2 and 3 are not the numbers I need in this case. On the other hand, 1 + 6 = 7, so I'll use 1 and 6:

  $x^2 + 7x + 6 = \mathbf{(x + 1)(x + 6)}$

Note that the order doesn't matter in multiplication, so the above answer could equally correctly be written as "$(x + 6)(x + 1)$".

- **Factor $x^2 – 5x + 6$.**

  The constant term is 6, but the middle coefficient this time is negative. Since I multiplied to a positive six, then the factors must have the same sign. (Remember that two negatives multiply to a positive.) Since I'm adding to a negative (–5), then both factors must be negative. So rather than using 2 and 3, as in the first example, this time I will use –2 and –3:

  $x^2 – 5x + 6 = \mathbf{(x – 2)(x – 3)}$

Note that you can use clues from the signs to determine which factors to use, as I did in this last example above:

- If c is positive, then the factors you're looking for are either both positive or else both negative.
  If b is positive, then the factors are positive
  If b is negative, then the factors are negative.
  In either case, you're looking for factors that add to b.

- If c is negative, then the factors you're looking for are of alternating signs;
  that is, one is negative and one is positive.
  If b is positive, then the larger factor is positive.
  If b is negative, then the larger factor is negative.
  In either case, you're looking for factors that are b units apart.

Let's try another one...

- **Factor $x^2 – 7x + 6$.**

  In this case, I am multiplying to a positive six, so the factors are either both positive or both negative. I am adding to a negative seven, so the factors are both negative. The factors of 6 that add up to 7 are 1 and 6, so I will use –1 and –6:

  $x^2 – 7x + 6 = \mathbf{(x – 1)(x – 6)}$

So far, "c" has always been positive. What if c is negative?

- **Factor $x^2 + x – 6$.**

  Since I am multiplying to a negative six, I need factors of opposite signs; that is, one factor will be positive and the other will be negative. The larger factor (in absolute value) will get the "plus" sign, because I am adding to a positive 1. Since these opposite-signed numbers will be adding to 1, I need the two factors to be one unit apart. The factor pairs for six are 1 and 6, and 2 and 3. The second pair are one apart, so I want to use 2 and 3, with the 3 getting the "plus" sign (so the 2 gets the "minus" sign).

  $x^2 + x – 6 = \mathbf{(x – 2)(x + 3)}$.

- **Factor $x^2 - x - 6$.**

  This looks just like the previous case, except that now the middle term is negative. I still want factors with opposite signs, and I still want factors that are one apart, but this time the larger factor gets the "minus" sign:

  $$x^2 - x - 6 = \mathbf{(x - 3)(x + 2)}$$

- **Factor $x^2 - 5x - 6$.**

  In this case, I still want factors of opposite signs, but now I want them to be five units apart, and the larger factor will get the "minus" sign. The factor pairs for six are 1 and 6, and 2 and 3. The first pair are five apart, so I'll use the numbers $+1$ and -6:

  $$x^2 - 5x - 6 = \mathbf{(x - 6)(x + 1)}$$

There is one special case, by the way, for factoring. Back when you were factoring plain old numbers, there were some numbers that didn't factor, such as 5 or 13. Recall that they are called "prime" numbers. The terminology is the same for polynomials:

- **Factor $x^2 + 7x - 6$**.

  Since the constant term is negative, I'll be needing a positive and a negative number such that, when I multiply them together, I get 6, but when I add them, I get 7. The factor pairs for 6 are 1 and 6, and 2 and 3. You may think that I should use 1 and 6, but —

  One of the factors has to be negative in order to multiply to get a "minus" six! Trying the first factor pair of 1 and 6, the sum would be either $(-1) + 6 = 5$ or else $1 + (-6) = -5$. And the other factor pair, 2 and 3, won't work, either, because $(-2) + 3 = 1$ and $2 + (-3) = -1$.

  In other words, there is no pair of factors of -6 that will add to $+7$. And if something isn't factorable, it's prime. Then $x^2 + 7x - 6$ is "**prime**", or "**unfactorable over the integers**" (because I couldn't find integers that would work).

Factoring Quadratics: The Hard Case: The Modified "a-b-c" Method, or "Box" *(page 2 of 4)*

Sections: The simple case, The hard case, The weird case

To factor a "hard" quadratic, we have to handle all three coefficients, not just the two we handled in the "easy" case, because the leading coefficient (the number on the x^2 term) is not 1. The first step in factoring will be to multiply "a" and "c"; then we'll need to find factors of the product "ac" that add up to "b".

- **Factor $2x^2 + x - 6$.**

  Looking at this quadratic, I have a = 2, b = 1, and c = –6, so ac = (2)(–6) = –12. So I need to find factors of –12 that add up to +1. The pairs of factors for 12 are 1 and 12, 2 and 6, and 3 and 4. Since –12 is negative, I need one factor to be positive and the other to be negative (because positve times negative is negative). This means that I'll want to use the pair "3 and 4", and I'll want the 3 to be negative, because –3 + 4 = +1. Now that I have found my factors, I will use what my students refer to as "box": I will draw a two-by-two grid, putting the first term in the upper left-hand corner and the last term in the lower right-hand corner, like this:

$2x^2$	
	-6

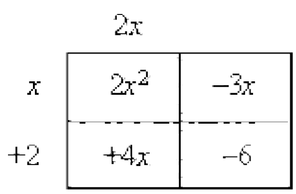
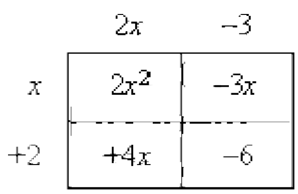
  Then I will take my factors –3 and 4 and put them, complete with their signs and variables, in the diagonal corners, like this:

$2x^2$	$-3x$
$+4x$	-6

  (It doesn't matter which way you do the diagonal entries; the answer will work out the same either way!)

  Then I'll factor the rows and columns like this:

from the top row			from the bottom row		
x	$2x^2$	$-3x$	x	$2x^2$	$-3x$
	$+4x$	-6	$+2$	$+4x$	-6

from the left column	from the right column
	

(Note: The signs for the bottom-row entry and the right-column entry come from the *closest* term that you are factoring from. Do not forget your signs!)

Now that I have factored the box, I can read off my answer from across the top and along the left-hand side:

$$2x^2 + x - 6 = \mathbf{(2x - 3)(x + 2)}.$$

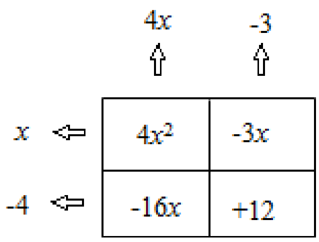
If your text or teacher factors "by grouping", you'll find that it is very easy to make mistakes with the signs. Using that method, you'll still have to find the numbers that add to the coefficient in the middle (the "3 and 4" in the above example), but your steps would look like this:

$$2x^2 + x - 6 = 2x^2 + 4x - 3x - 6 = 2x(x + 2) - 3(x + 2) = (x + 2)(2x - 3)$$

You would get the same answer, but my students have always found "box" to be easier and more reliable, especially in cases like the one above where you're having to try to keep track of "minus" signs. In my experience, students using "by grouping" get in the habit of dropping the sign in the middle ("isn't it always a 'plus' sign?") and generally forget to factor the "minus" sign out of the second "group" correctly. (In this example, the student would either have gotten factors of "$x + 2$" and "$x - 2$", and been stuck, or else would have factored as "$(x + 2)(2x + 3)$". Either way, he would have gotten the wrong answer.) It was this continual confusion that led me to switch from factoring "by grouping" to using "box". My students just do better with "box", but you should use whatever works best for you.

- **Factor $4x^2 - 19x + 12$.**

  The coefficients are a = 4, b = –19, and c = 12, so ac = 48. Since 48 is positive, I need two factors that are either both positive or else both negative (positive times positive is positive, and negative times negative is positive). Since –19 is negative, I need the factors both to be negative. The pairs of factors for 48 are 1 and 48, 2 and 24, 3 and 16, 4 and 12, and 6 and 8. Since –3 + (–16) = –19, I will use –3 and –16:



(Note: The above graphic is animated in the original ("live") web lesson.)

$4x^2 - 19x + 12 =$ **$(x - 4)(4x - 3)$**.

- **Factor $5x^2 - 10x + 6$**

  I have a = 5, b = –10, and c = 6, so ac = +30. Since ac is positive and b is negative, I need to find two factors that are both negative and which add up to –10. But the pairs of factors for 30 are 1 and 30, 2 and 15, 3 and 10, and 5 and 6. None of these pairs adds to 10.

  Then this quadratic is said to be "**unfactorable over the integers**" (because I couldn't find integer factors that worked), or it might be called "**prime**". The specific terminology you should use will probably depend upon your text. If in doubt, ask your teacher what terminology you should use to refer to unfactorable quadratics.

The following example illustrates the fact that "box" works *only* if you have first removed all common factors.

- **Factor $2x^2 - 4x - 16$**

  If I don't first take out the common factor of "2", I'll find the factors of $2 \times (-16) = -32$ that add to –4, which are –8 and +4. Doing "box", I then get:

	$2x$	-8
$2x$	$2x^2$	$-8x$
$+4$	$+4x$	-16

  In other words, I'll have factored as $(2x - 8)(2x + 4)$. But when I multiply this back out, I'll get $4x^2$ for the leading term, instead of $2x^2$. By not taking that common factor out first, I'll have managed to get "extra" factors in "box"; in particular, I'll have gotten the wrong answer. To do things properly:

  First, I need to remove the common factor of 2 from each term, to get $2x^2 - 4x - 16 = 2(x^2 - 2x - 8)$. Then I need to factor the remaining quadratic: $x^2 - 2x - 8 = (x - 4)(x + 2)$. I need to be careful not to forget the factored-out "2" when I write down my final answer:

  $2x^2 - 4x - 16 =$ **$2(x - 4)(x + 2)$**.

Factoring Quadratics: The Hard Case: Examples of How to Use "Box" *(page 3 of 4)*

Sections: The simple case, The hard case, The weird case

The one special case that often causes students some trouble is when the leading coeffiecient is a negative one. A good first step is to factor out the -1.

- **Factor $-6x^2 - x + 2$**

  I will first take out the minus one to get $-6x^2 - x + 2 = -1(6x^2 + x - 2)$. (I need to remember that *every* sign changes when I multiply or divide by a negative. I mustn't fall into the trap of taking the –1 out of only the first term; I must take it out of all three!) Factoring the contents of the parentheses then gives me:

	$3x$	$+2$
$2x$	$6x^2$	$+4x$
-1	$-3x$	-2

  Then:

  $$-6x^2 - x + 2 = -1(6x^2 + x - 2) = \mathbf{-1(2x - 1)(3x + 2)}$$

Putting these two techniques together (factoring out anything common, and taking out a leading negative sign), you can handle such problems as:

- **Factor $-6x^2 + 15x + 36$**

  First, I will first remove the common factor of 3, taking the leading negative sign with it: $-6x^2 + 15x + 36 = -3(2x^2 - 5x - 12)$. Then I'll factor the remaining quadratic: $2x^2 - 5x - 12 = (x - 4)(2x + 3)$:

	x	-4
$2x$	$2x^2$	$-8x$
$+3$	$+3x$	-12

  When I write down my answer, I need to remember to include the -3 factor:

  $$-6x^2 + 15x + 36 = \mathbf{-3(x - 4)(2x + 3)}.$$

A disguised version of this factoring-out-the-negative case is when they give you a backwards quadratic where the squared term is subtracted. For example, if they give you something like $6 + 5x + x^2$, you would just reverse the quadratic to put it back in the "normal" order, and then factor: $6 + 5x + x^2 = x^2 + 5x + 6 = (x + 2)(x + 3)$. You can do this because order doesn't matter in addition. In subtraction, however, order *does* matter, and you need to be careful with signs.

- **Factor $6 + x - x^2$**

  First, I will want to reverse the quadratic, but I'll need to take care with the signs:
  $6 + x - x^2 = -x^2 + x + 6$. Then I'll factor out the -1, and factor the remaining quadratic as usual:
  $-x^2 + x + 6 = -1(x^2 - x - 6) = \mathbf{-1(x + 2)(x - 3)}$

Sometimes the numbers in a factorization are large enough that the factor pair you need is hard to find. But if you list all the factor pairs, in order, you will eventually find the pair you need.

- **Factor $20x^2 - 17x - 63$**

  Multiplying a and c, I get $(20)(-63) = -1260$. Off the top of my head, I have no idea what factors I'll need to use. All I know so far is that those factors will have opposite signs, and that they'll be seventeen units apart. So I'll make a list of factor pairs, and see where that leads. (My calculator can be very helpful for this!)

factor pairs		the differences
1, 1260	As you can see (to the left), I get a very long list of factor pairs.	1260 – 1 = 1259
2, 630		630 – 2 = 628
3, 420		420 – 3 = 417
4, 315		315 – 4 = 311
5, 252	Now that I have my list of factor pairs, I can subtract the pairs to find the differences. If there is a pair of factors with a difference of 17, then I can factor the quadratic. If not, then I will know that the quadratic is prime.	252 – 5 = 247
6, 210		210 – 6 = 204
7, 180		180 – 7 = 173
9, 140		140 – 9 = 131
10, 126		126 – 10 = 116
12, 105		105 – 12 = 93
14, 90		90 – 14 = 76
15, 84		84 – 15 = 69
18, 70		70 – 18 = 52
20, 63		63 – 20 = 43
21, 60		60 – 21 = 39
28, 45	As you can see (to the right), there is one pair of factors that suits my needs; namely, 45 and 28.	**45 – 28 = 17**
30, 42		42 – 30 = 12
35, 36		36 – 35 = 1

  Now that I have my factor pair (with the larger number having the "minus" sign), I can factor the quadratic:

	$4x$	-9
$5x$	$20x^2$	$-45x$
$+7$	$+28x$	-63

$20x^2 - 17x - 63 =$ **$(4x - 9)(5x + 7)$**

By the way, you should *expect* an exercise as long as this on the next test. Don't waste a lot of time trying to "eyeball" the solution; when you have numbers this big, it is actually faster to write down the list of factor pairs.

There is one other type of quadratic that looks kind of different, but the factoring works in exactly the same way:

- **Factor $6x^2 + xy - 12y^2$.**

This may look bad, what with the y^2 at the end, but it factors just like all the ones above. Remember, from the "simple" case, that we knew that the factors had to be of the form:

(x + something)(x + something else)

...because we knew we'd multiplied factors that looked like this in order to get the quadratic in the first place. This was how we knew that we needed x's in the fronts of our parentheses. In the same way, we know that we must have multiplied factors of the form:

(an x term + a y term)(another x term + another y term)

...to get that y^2 term at the end of the quadratic. So we'll need to put y's at the ends of our parentheses. But other than this, the process will work as usual.

First, I need to find factors of $(6)(-12) = -72$ that add to $+1$; I'll use $+9$ and -8. Then "box" gives me:

	$3x$	$-4y$
$2x$	$6x^2$	$-8xy$
$+3y$	$+9xy$	$-12y^2$

So $6x^2 + xy - 12y^2$ factors as **$(2x + 3y)(3x - 4y)$**.

Quadratic factoring can pop up in even more exotic forms than the last example above....

Factoring Quadratics: The Weird Case *(page 4 of 4)*

Sections: The simple case, The hard case, The weird case

This is the case where it doesn't seem like you're factoring a quadratic, but you really are. You'll need to be clever with these, but they reduce to little more than pattern-recognition once you catch on to how to do them.

- **Factor $x^4 - 2x^2 - 8$.**

  At first glance, this does not appear to be a quadratic. But it is — sort of. Taking a closer look at those exponents, I see that the power on the leading term is "4", and the power on the middle term is "2", which is half of 4. With a "regular" quadratic, I would have the powers 2 and 1, where 1 is half of 2. So this polynomial follows the quadratic pattern: it is a quadratic "in x^2", rather than "in x". In fact, I can rewrite the expression as:

  $$(x^2)^2 - 2(x^2) - 8$$

  My leading coefficient is just 1, so this is an "easy" quadratic. I'll draw my parentheses as usual, but I'll put an x^2 at the beginning of each factor:

  $$(x^2 \quad)(x^2 \quad)$$

  Factors of –8 that add to –2 are –4 and +2 , so I get:

  $$(x^2 - 4)(x^2 + 2)$$

  I still need to check to see if there is any further factoring possible. In this case, I still have a difference of squares, $x^2 - 4$, that I can factor:

  $$x^4 - 2x^2 - 8 = (x^2 - 4)(x^2 + 2) = \mathbf{(x - 2)(x + 2)(x^2 + 2)}$$

Some books will encourage you to switch variables, plugging in a "y" for the "x^2", so you'd get:

$$y^2 - 2y - 8$$

This is clearly a quadratic, which you can factor as usual:

$$y^2 - 2y - 8 = (y - 4)(y + 2)$$

Then you'd plug the "x^2" back in for the "y" to get:

$x^4 - 2x^2 - 8$
$= (x^2)^2 - 2(x^2) - 8$
$= y^2 - 2y - 8$
$= (y - 4)(y + 2)$
$= (x^2 - 4)(x^2 + 2)$

It is not necessary to do that substitution of "y" for "x^2"; some students find it helpful, but many find it confusing. You should do whatever works better for you.

What follows are some fairly typical examples of "weird" quadratics. Note that there are always three terms in the expression (or two, if it's a difference of squares), and the power (or exponent) on the middle term is always half of the power on the leading term.

- **Factor $x^6 + 6x^3 + 5$.**

  The power on the middle term is 3, which is half of the power on the leading term, so this is a quadratic in x^3. I will factor as usual:

  $x^6 + 6x^3 + 5$
  $= (x^3)^2 + 6(x^3) + 5$
  $= (x^3 + 5)(x^3 + 1)$

  Since the second of these factors is a sum of cubes, I can factor further:

  $x^3 + 1 = (x)^3 + (1)^3 = (x + 1)(x^2 - x + 1)$

  Then the complete answer is:

  $x^6 + 6x^3 + 5 = \mathbf{(x^3 + 5)(x + 1)(x^2 - x + 1)}$

In these two examples, after I'd factored the "quadratic", I still had to do some more factoring. This will not always be the case, but will generally be the case on tests when the instructor will be seeing if you're on top of your game. So keep in mind that, just because you've done one factoring step, this doesn't mean that you're done with the exercise.

- **Factor $x^{2/3} - x^{1/3} - 6$.**

  The power on the middle term is $^1/_3$, which is half of the power on the leading term, so this is a quadratic in $x^{1/3}$. I will factor as usual:

  $x^{2/3} - x^{1/3} - 6$
  $= (x^{1/3})^2 - 1(x^{1/3}) - 6$
  $= \mathbf{(x^{1/3} - 3)(x^{1/3} + 2)}$

- **Factor $x + 5x^{1/2} + 4$.**

  The power on the middle term is $^1/_2$, which is half of the power on the leading term, so this is a quadratic in $x^{1/2}$. I will factor as usual:

  $$\begin{aligned} &x + 5x^{1/2} + 4 \\ &\quad = (x^{1/2})^2 + 5(x^{1/2}) + 4 \\ &\quad = \mathbf{(x^{1/2} + 4)(x^{1/2} + 1)} \end{aligned}$$

- **Factor $4x^4 – 25$.**

  This is just a difference of squares. I will factor as usual:

  $$\begin{aligned} &4x^4 – 25 \\ &\quad = (2x^2)^2 – (5^2) \\ &\quad = \mathbf{(2x^2 – 5)(2x^2 + 5)} \end{aligned}$$

- **Factor $(x – 3)^4 + 2(x – 3)^2 – 8$.**

  The power on the middle term is 2, which is half of the power on the leading term, so this is a quadratic in $(x – 3)^2$. I will factor as usual:

  $$\begin{aligned} &(x – 3)^4 + 2(x – 3)^2 – 8 \\ &\quad = ((x – 3)^2)^2 + 2(x – 3)^2 – 8 \\ &\quad = y^2 + 2y – 8 \\ &\quad = (y + 4)(y – 2) \\ &\quad = ((x – 3)^2 + 4)((x – 3)^2 – 2) \\ &\quad = ((x^2 – 6x + 9) + 4)((x^2 – 6x + 9) – 2) \\ &\quad = (x^2 – 6x + 13)(x^2 – 6x + 7) \end{aligned}$$

  Since neither of these quadratics factors, I am done.

  $$(x – 3)^4 + 2(x – 3)^2 – 8 = \mathbf{(x^2 – 6x + 13)(x^2 – 6x + 7)}$$

You can check the answer this last exercise by multiplying the two polynomial factors, and then multiplying out the original expression and verifying that they both simplify to the same fourth-degree polynomial: $x^4 – 12x^3 + 56x^2 – 120x + 91$.